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s-wave bound and scattering state wave functions for a velocity-dependent Kisslinger potential

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Abstract. A relation linking the normalized s-wave scattering and the corresponding bound state wave functions at bound state poles is derived. This is done in the case of a non-local, velocity-dependent Kisslinger potential. Using formal scattering theory, we present two analytical proofs of the validity of the theorem. The first tackles the non-local potential directly, while the other transforms the potential to an equivalent local but energy-dependent one. The theorem is tested both analytically and numerically by solving the Schrödinger equation exactly for the scattering and bound state wave functions when the Kisslinger potential has the form of a square well. A first order approximation to the deviation from the theorem away from bound state poles is obtained analytically. Furthermore, a proof of the analyticity of the Jost solutions in the presence of a non-local potential term is also given.

PACS. 03.65.Nk Scattering theory – 24.90.+d Other topics in nuclear reactions: general

1 Introduction

The extrapolation of the scattering wave functions to negative energies corresponding to the bound state poles has been carried out in previous works [1–3]. Goldberger and Watson [1], and Joachian [2] have derived a relation linking the normalized scattering and the corresponding bound state wave functions at bound state poles. They showed that the relative normalization, which is the ratio of the normalized bound and scattering wave functions at a bound state pole, depends on the details of the potential through the corresponding Jost functions and their derivatives. A more recent work [3], concluded that the relative normalization does not depend on the details of the potential but is uniquely given by the bound state binding energy. This was done for the case of a local potential. The case of a non-local but separable Yamaguchi potential was also studied and the relative normalization was found to be uniquely determined by the binding energy, provided the binding energy is small [4]. Such a theorem would be useful in final-state interaction theory. For example, it allows one to predict the cross-section for pp $\to$ pnπ^+ in terms of those for pp $\to$ dπ^+ and pp $\to$ ppπ^0 at low energies [5,6].

In this work we have extended the theorem to the case of non-local, velocity-dependent Kisslinger potential in the s-wave case. Following closely the method developed in [3], we present two analytical proofs of the theory. The first deals with the non-local potential directly, while the other begins by transforming the potential into a local but energy-dependent one. We also derive an analytical form for the deviation from the theorem away from bound state poles.

The velocity-dependent potential was introduced to describe the scattering of particles off complex nuclei [7]. It was designed to provide a kind of optical potential for pion-nucleus scattering, and took account of the predominantly p-wave nature of the elementary pion-nucleon coherent scattering. Kisslinger theory resulted in a term proportional to

$$\nabla \cdot (\rho \nabla \psi) = \rho \nabla^2 \psi + \nabla \rho \cdot \nabla \psi. \tag{1.1}$$

The first term on the right is proportional to the kinetic energy and combines with the kinetic energy term in the wave equation. The second term is proportional to the rate of the change of density, thus it is sensitive to the diffuse edge in nuclei. Consequently, Kisslinger potential was able to predict the large backward scattering in the scattering of mesons by light nuclei, where the effect of a diffuse edge is important. A velocity-dependent potential was also used to replace the hard-core nucleon-nucleon potential in order to explain the isotropy of the p-p differential cross-section at 100 MeV [8].

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2 Non-local velocity-dependent Kisslinger potential

The non-local velocity-dependent potential $V(r)$ derived by Kisslinger [7] may be written as

$$2mV(r) = U(r) + \nabla \cdot (\rho(r)\nabla), \tag{2.1}$$

where the reduced potential $U(r)$ and the non-local term $\rho(r)$ are real, spherically symmetric functions of the radial variable r. For a particle of mass m and energy $E = k^2/2m$, the s-wave Schrödinger equation including the non-local potential may be written as

$$(1-\rho(r))\frac{\mathrm{d}^2 v(k,r)}{\mathrm{d}r^2} - \left[\frac{\mathrm{d}v(k,r)}{\mathrm{d}r} - \frac{v(r)}{r}\right]\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} \times k^2 v(k,r) = U(r)v(k,r). \tag{2.2}$$

We shall start by deriving the conditions that $U(r)$ and $\rho(r)$ must satisfy such that the above equation has well-behaved, physically acceptable solutions.

2.1 Small r behavior

Equation (2.2) may be written in the form (the dependence of ρ on r is suppressed for clarity of presentation):

$$v''(k,r) - \frac{\rho'}{(1-\rho)}v'(k,r) \times \frac{1}{(1-\rho)}\left[k^2 + \frac{\rho'}{r} - U(r)\right]v(k,r) = 0, \tag{2.3}$$

which, in the standard form, is expressed as

$$v''(k,r) + P(r)v'(k,r) + Q(r)v(k,r) = 0. \tag{2.4}$$

For (2.3) to be regular at the origin, we require:

$$\lim_{r\to 0} rP(r) < \infty, \qquad \lim_{r\to 0} r^2 Q(r) < \infty. \tag{2.5}$$

It is clear from (2.3) that $\rho(r)$ must not equal to the value of 1. So for all r either $\rho(r) < 1$ or $\rho(r) > 1$.

In the vicinity of the origin, we may assume the local and non-local parts of the potential to behave as

$$U(r) \approx c_0\, r^q, \qquad \rho(r) \approx b_0\, r^p. \tag{2.6}$$

If $p > 0$, it follows that $\rho(0) = 0$ and $\rho(r) < 1$ for all r. In this case eq. (2.3) is regular at the origin provided $q \geq -2$. Using the expansion

$$v(k,r) = \sum_{n=0}^{\infty} a_n r^{n+s}, \tag{2.7}$$

and (2.6), eq. (2.3) reads

$$\sum_n a_n(n+s)(n+s-1)r^{n+s-2} - pb_0\sum_n a_n s r^{n+s+p-2} + k^2\sum_n a_n r^{n+s} - c_0\sum a_n r^{n+s+q} = 0. \tag{2.8}$$

For $q > -2$ the corresponding indecial equation is

$$s(s-1) = 0. \tag{2.9}$$

According to the theorem of Frobenius, we obtain the series solution $v(k,r) \approx r$ near the origin. Hence, we may impose the boundary condition

$$v(k,0) = 0. \tag{2.10}$$

For $q = -2$ if we consider the coefficient of r^{s-2} it is easy to see that physical solutions can only be obtained if $c_0 > -1/4$ [1].

In the case $p < 0$ then $\rho(r)$ is singular at the origin. If $\rho(r)$ is repulsive near $r = 0$ ($b_0 > 0$), then $\rho(r) > 1$ for all r. However, if $b_0 < 0$ near the origin then the non-local part is attractive close to the origin and $\rho(r) < 1$ for all r. In either case, the term $1 - \rho(r) \approx -b_0 r^p$ close to $r = 0$, and eq. (2.3) is regular at the origin provided $q - p \geq -2$. We then obtain

$$\sum_n (n+s+p)(n+s-1)a_n r^{n+s-2} - \frac{k^2}{b_0}\sum_n a_n r^{n+s-p} + \frac{c_0}{b_0}\sum_n a_n r^{n+s+q-p} = 0. \tag{2.11}$$

If we choose the minimum value $q - p = -2$ and consider the coefficient of r^{s-2}, it is not difficult to see that for (2.3) to have physically acceptable solutions, which are regular at $r = 0$, then the ranges of b_0 and c_0 are limited according to

$$\frac{c_0}{b_0} \leq \frac{1}{4}(p+1)^2. \tag{2.12}$$

However, in order to have wider applicability we shall assume $q - p > -2$. In this domain the indecial equation of (2.11) reads

$$s^2 + (p-1)s - p = 0. \tag{2.13}$$

Consequently, in the vicinity of the origin $v(k,r) \approx r$ or r^{-p}. And therefore, by the theory of Frobenius, we get at least one series solution of the form (2.7) corresponding to the larger value of s. Therefore, we may impose the boundary condition given in (2.10). In sect. 3 we shall return to the allowed values of p.

2.2 Large r behavior

To determine the behaviors of the terms $U(r)$ and $\rho(r)$ at large distances, we write $v(k,r) = g(r)e^{ikr}$ and assume $g(r)$ to be a slowly varying function. Substituting for $v(k,r)$ in (2.2) and ignoring the small term $g''(r)$ results in

$$\ln(g(r)) = \int_b^{\infty} \frac{U - \rho k^2 + (ik - 1/r)\rho'}{2ik(1-\rho) - \rho'}\mathrm{d}r', \tag{2.14}$$

where $b > 0$. For $g(r)$, and hence $v(k,r)$, to be finite we require

$$\lim_{r\to\infty} U(r) = \frac{M}{r^{1+\epsilon}}, \tag{2.15}$$

and

$$\lim_{r\to\infty} \rho(r) = \frac{N}{r^{1+\epsilon}}, \tag{2.16}$$

where $\epsilon > 0$, and M, N are finite constants. That is at large distances both parts of the potential must fall off faster than $1/r$.

3 Analyticity of the Jost solutions

As shown in Appendix A, the reduced scattering wave function $v(k,r)$ may be expressed as a linear combination of Jost solutions and functions *viz.*,

$$v(k,r) = -\frac{1}{2ik}\frac{1}{\sqrt{1-\rho(r)}}\frac{1}{|f(k)|} \times \left[f(-k)f(k,r) - f(k)f(-k,r)\right], \tag{3.1}$$

where the Jost solutions behave asymptotically as

$$\lim_{r\to\infty} f(k,r) = e^{-ikr}, \qquad \lim_{r\to\infty} f(-k,r) = e^{ikr}. \tag{3.2}$$

The Jost functions are defined as

$$f(\pm k) = f(\pm k, 0). \tag{3.3}$$

The analyticity of the Jost solutions in the presence of the non-local term $\rho(r)$ is proved in Appendix A, provided $\rho(r)$ satisfies the following conditions:

$$\int_0^\infty r\,|\rho''(r)|\mathrm{d}r < \infty, \qquad \int_0^\infty r^2\,|\rho(r)|\mathrm{d}r < \infty. \tag{3.4}$$

The first condition implies that $\rho'(r)$ diverges less than $1/r$ for small r. This condition is satisfied if in the vicinity of the origin $\rho(r) \approx b_0 r^p$ where $p > 0$. However, the second condition demands that $\rho(r)$ falls off faster than $1/r^3$ at large distances. As for the local part, it should satisfy

$$\int_0^\infty r\,|U(r)|\mathrm{d}r < \infty, \qquad \int_0^\infty r^2\,|U(r)|\mathrm{d}r < \infty. \tag{3.5}$$

Those are the same conditions in the standard case where $\rho(r) = 0$, suggesting that $U(r)$ diverges slower than $1/r$ for small r and falls off faster than $1/r^3$ at infinity.

The boundary conditions in (3.2) define $f(k,r)$ in the lower half of the complex k-plane, while $f(-k,r)$ is defined in the upper half and each is analytic in the region over which it is defined. The analyticity of the Jost solutions may be extended if we impose the conditions

$$\int_0^\infty \mathrm{d}r\, e^{mr}\,|U(r)| < \infty, \quad \int_0^\infty \mathrm{d}r\, e^{mr}\,|\rho(r)| < \infty, \tag{3.6}$$

where m is real and positive. Then $f(k,r)$ is analytic for $\mathrm{Im}(k) < m/2$, while $f(-k,r)$ is analytic for $\mathrm{Im}(k) > -m/2$. It follows from the boundary conditions (3.2) and the form of eq. (2.2) that in the region of analyticity, including the real axis, the Jost solutions and functions satisfy the conditions

$$f^*(-k^*, r) = f(k,r), \tag{3.7}$$

and

$$f^*(-k^*) = f(k). \tag{3.8}$$

The symmetry properties of the Jost solutions and functions stated above imply that $v(k,r)$ in (3.1) is real for real values of k. The scattering function may be expressed in the form

$$v(k,r) = -\frac{1}{2ik}\frac{1}{\sqrt{1-\rho(r)}} \times \left[e^{-i\delta_0(k)}f(k,r) - e^{i\delta_0(k)}f(-k,r)\right], \tag{3.9}$$

where $\delta_0(k)$ is the s-wave scattering phase shift. In terms of $\delta_0(k)$ the scattering matrix $S(k)$ is defined as

$$S(k) = \frac{f(k)}{f(-k)} = e^{2i\delta_0(k)}. \tag{3.10}$$

Using (3.2) and the fact that $\rho(r)$ and $U(r)$ both vanish faster than $1/r$ at infinity, then $v(k,r)$ behaves asymptotically as

$$v(k,r) = \frac{1}{k}\sin(kr + \delta_0(k)). \tag{3.11}$$

As explained in Appendix A, provided the boundary conditions in (3.6) hold, $v(k,r)$ can be analytically continued from the real axis into the upper half of the complex k-plane. The zeros of the Jost function $f(-k)$ situated on the positive part of the imaginary axis are simple, and are poles of the scattering matrix $S(k)$, corresponding to the positions of bound states. Indeed, for $k = i\lambda$ with $\lambda > 0$, using (3.1), we have

$$v(i\lambda,r) = -\frac{1}{2\lambda}\frac{1}{\sqrt{1-\rho(r)}}\frac{1}{|f(i\lambda)|}f(i\lambda)f(-i\lambda,r) = -\frac{1}{2\lambda}\frac{1}{\sqrt{1-\rho(r)}}e^{i\delta_0(i\lambda)}f(-i\lambda,r), \tag{3.12}$$

which at infinity behaves as

$$v(i\lambda,r) = -\frac{1}{2\lambda}\frac{1}{|f(i\lambda)|}f(i\lambda)e^{-\lambda r} = -\frac{1}{2\lambda}e^{i\delta_0(i\lambda)}e^{-\lambda r}. \tag{3.13}$$

The function $v(i\lambda,r)$ vanishes at the origin and decreases exponentially at large distances. Hence, it is a square integrable function.

For bound states, the s-wave Schrödinger equation including the non-local term is

$$(1-\rho(r))\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2} - \left[\frac{\mathrm{d}u(r)}{\mathrm{d}r} - \frac{u(r)}{r}\right]\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} - \lambda^2 u(r) = U(r)u(r), \tag{3.14}$$

where $u(r)$ is the reduced bound state wave function satisfying the boundary conditions

$$u(0) = 0, \qquad \lim_{r\to\infty} u(r) \to e^{-\lambda r}. \tag{3.15}$$

In the vicinity of an isolated bound state pole it is possible to parameterize $S(k)$ as

$$S(k) = e^{2i\delta_0(k)} = \frac{[NG(k)]^2}{\lambda + ik}, \tag{3.16}$$

where N^2 is the residue at the bound state pole. In the unitarized scattering length approximation $N^2 = 2\lambda$ and

$$G^2(k) = \frac{\lambda - ik}{2\lambda}. \tag{3.17}$$

In general $G(k)$ is an analytic function of k in the vicinity of $k = i\lambda$ with the condition $G(i\lambda) = 1$.

The boundary conditions imposed on the scattering and bound state wave functions at the origin and at infinity, combined with the reality of $U(r)$ and $\rho(r)$, ensure that, for k real, $v(k,r)$ and $u(r)$ remain real for all values of r. The scattering wave function, $v(k,r)$, can then be analytically continued in k to the position of a bound state pole $k = i\lambda$ on the positive imaginary axis. However, special attention must be given to the singularity structure of the factor $e^{i\delta_0(k)}$ as it has a branch cut at the position of the pole. Making use of eq. (3.16) at the position of a bound state pole, we get

$$\left[\sqrt{2\lambda(\lambda^2+k^2)}\, v(k,r)\right]_{k=i\lambda} = -\left[\sqrt{\lambda+ik}\, e^{i\delta_0(k)}\right]_{k=i\lambda} f(-i\lambda, r) = -Nu(r). \tag{3.18}$$

Now our aim is to show that N is uniquely determined by the normalization of the bound state wave function. Multiplying (2.2) by $u(r)$ and (3.14) by $v(k,r)$ and then rearranging, gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{(1-\rho(r))\left[u'(r)v(k,r) - u(r)v'(k,r)\right]\right\} = (\lambda^2+k^2)v(k,r)u(r). \tag{3.19}$$

Integrating the above equation, and noting that both $u(r)$ and $v(k,r)$ vanish at the origin, results in

$$(1-\rho(r))\left[u'(r)v(k,r) - u(r)v'(k,r)\right] = (\lambda^2+k^2)\int_0^r v(k,r')u(r')\mathrm{d}r'. \tag{3.20}$$

As $r \to \infty$ both sides of the above equation vanish. To avoid this, define

$$w(k,r) = 2ik\sqrt{\lambda+ik}\, v(k,r), \tag{3.21}$$

which by eq. (3.16) has the limit at the pole

$$w(i\lambda, r) = Nu(r). \tag{3.22}$$

Differentiating the resulting

$$(1-\rho(r))\left[u'(r)w(k,r) - u(r)w'(k,r)\right] = (\lambda^2+k^2)\int_0^r u(r')w(k,r')\mathrm{d}r', \tag{3.23}$$

with respect to k, leads to

$$(1-\rho(r))\left[u'(r)\dot{w}(k,r) - u(r)\dot{w}'(k,r)\right] = \int_0^r \left[(\lambda^2+k^2)\dot{w}(k,r') + 2kw(k,r')\right]u(r')\mathrm{d}r', \tag{3.24}$$

where the prime refers to differentiation with respect to r, while the dot is that with respect to k. By taking the limit $k \to i\lambda$ followed by $r \to \infty$, the first term in the integrand vanishes and the right-hand side becomes

$$2i\lambda N \int_0^r u^2(r')\mathrm{d}r'. \tag{3.25}$$

Using eqs. (3.9), (3.16) and (3.21) in the vicinity of the pole leads to

$$\sqrt{1-\rho(r)}w(k,r) = N\,G(k)\,f(-k,r) - \frac{(\lambda+ik)}{NG}f(k,r), \tag{3.26}$$

which upon differentiation with respect to k gives

$$\sqrt{1-\rho(r)}\dot{w}(k,r) = N\dot{G}f(-k,r) + NG\dot{f}(-k,r) - \frac{i}{NG}f(k,r) \tag{3.27}$$

$$-\frac{(\lambda+ik)}{NG}\dot{f}(k,r) + (\lambda+ik)\frac{\dot{G}}{NG^2}f(k,r). \tag{3.28}$$

At the position of a bound state pole, $G(i\lambda) = 1$, consequently

$$w(i\lambda, r) = \frac{1}{\sqrt{1-\rho(r)}}\left[Nf(-i\lambda, r)\right] \longrightarrow Ne^{-\lambda r}. \tag{3.29}$$

and

$$\dot{w}(i\lambda, r) = \frac{1}{\sqrt{1-\rho}}\left[N\dot{G}f(-i\lambda,r) + N\dot{f}(-i\lambda,r) - \frac{i}{N}f(i\lambda,r)\right] \tag{3.30}$$

$$\longrightarrow -\frac{i}{N}e^{\lambda r}, \tag{3.31}$$

where the limit as $r \to \infty$ was taken in the last two equations. Since the bound state wave function decreases exponentially with r, the only surviving term on the left-hand side of eq. (3.24) is that proportional to $f(i\lambda, r)$. Equating the two sides of eq. (3.24), and noting that $\rho(r) \to 0$ for large r, results in

$$1 = N^2\int_0^\infty u^2(r)\mathrm{d}r. \tag{3.32}$$

This proves that N is actually the normalization constant of the bound state wave function. The last result may equivalently be restated as

$$\lim_{k\to i\lambda}\left[\sqrt{2\lambda(\lambda^2+k^2)}\,v(k,r)\right] = -u_{\rm n}\ , \qquad (3.33)$$

where $u_{\rm n}$ is the normalized bound state wave function.

4 The equivalent local energy-dependent potential

It is a well-known feature of the Kisslinger potential that it may be transformed into a local but energy-dependent potential through the transformation on the wave function [9]

$$v(k,r) = \frac{\chi(k,r)}{\sqrt{1-\rho(r)}}. \qquad (4.1)$$

Substituting the above in the Schrödinger eq. (2.2) leads to

$$\frac{{\rm d}^2\chi(k,r)}{{\rm d}r^2} + \left[\frac{\rho''(r)}{2} - \frac{(\rho'(r))^2}{4(1-\rho(r))} + \frac{\rho'(r)}{r} + k^2 - U(r)\right]\frac{\chi(k,r)}{1-\rho(r)} = 0. \qquad (4.2)$$

This has the same form of a Schrödinger equation with an effective energy-dependent potential term $\rho(r)k^2/(1-\rho(r))$. As shown in Appendix A, $\chi(k,r)$ may be expressed in terms of a linear combination of Jost solutions and functions. The conditions that $U(r)$ and $\rho(r)$ have to satisfy in order that the Jost solutions are analytic in the appropriate range are discussed in the Appendix.

For bound states $k = i\lambda,\ \lambda > 0$, the corresponding equation is

$$\frac{{\rm d}^2\phi(r)}{{\rm d}r^2} + \left[\frac{\rho''(r)}{2} - \frac{(\rho'(r))^2}{4(1-\rho(r))} + \frac{\rho'(r)}{r} - \lambda^2 - U(r)\right]\frac{\phi(r)}{1-\rho(r)} = 0, \qquad (4.3)$$

where $\phi(r)$ is the bound state wave function transformed according to

$$u(r) = \frac{\phi(r)}{\sqrt{1-\rho(r)}}. \qquad (4.4)$$

The functions $\chi(k,r)$ and $\phi(r)$ satisfy the boundary conditions

$$\chi(k,r) = \phi(r) = 0, \qquad (4.5)$$

and

$$\lim_{r\to\infty}\chi(k,r) = \frac{1}{k}\sin(kr+\delta_0(k)), \quad \lim_{r\to\infty}u(r) = e^{-\lambda r}. \qquad (4.6)$$

The above boundary conditions and the reality of $U(r)$ and $\rho(r)$ ensure that both $\chi(k,r)$ and $\phi(r)$ are real for all values of r, provided k is real. Therefore, $\chi(k,r)$ may be analytically continued from the real axis into the upper half of the complex k-plane.

Manipulation of eqs. (4.2) and (4.3) leads to

$$\frac{\rm d}{{\rm d}r}\left[\chi(k,r)\phi'(r) - \chi'(k,r)\phi(r)\right] = (\lambda^2+k^2)\frac{\chi(k,r)\ \phi(r)}{(1-\rho(r))}. \qquad (4.7)$$

Noting that both $\chi(k,r)$ and $\phi(r)$ vanish at the origin the above equation may be integrated to give

$$\chi(k,r)\phi'(r) - \chi'(k,r)\phi(r) = (\lambda^2+k^2)\int_0^r \frac{\chi(k,r')\ \phi(r')}{(1-\rho(r'))}{\rm d}r'. \qquad (4.8)$$

Defining

$$w(k,r) = 2ik\sqrt{\lambda+ik}\ \chi(k,r) \qquad (4.9)$$

and proceeding in the same way as we did in the previous section one arrives at

$$1 = N^2\int_0^\infty \left(\frac{\phi}{\sqrt{1-\rho}}\right)^2 {\rm d}r'. \qquad (4.10)$$

It is worth noting that it is the bound state wave function $u(r) = \phi(r)/\sqrt{1-\rho(r)}$ which is normalized to unity not $\phi(r)$ itself. Thus once again we have shown that N is actually the normalization factor of the bound state wave function.

5 Square well Kisslinger potential

The exact solutions of the Schrödinger equation using a square well Kisslinger potential provide a valuable clue as to the range of usefulness of the theorem in the non-local case. Our ansatz for the non-local and local parts of the potential respectively are

$$\rho(r) = A\ \theta(a-r), \qquad (5.1)$$

while the local part of the potential is taken to be

$$U(r) = -U_o\ \theta(a-r), \qquad (5.2)$$

where a is the common radius of both potentials. The boundary conditions are such that the wave functions must be continuous at $r = a$. But the derivatives are not due to the effect of the non-local term $\rho(r)$ at the sharp boundary. By integrating the Schrödinger equation from $a-\epsilon$ to $a+\epsilon$, the condition on the derivatives is

$$(1-A)\psi'(r<a) = \psi'(r>a). \qquad (5.3)$$

It is not difficult to evaluate the s-wave scattering and the corresponding bound state wave functions in this case. Substitution of the scattering wave function on the left-hand side of eq. (3.33), and using La Hospital's rule, one readily recovers the right-hand side.

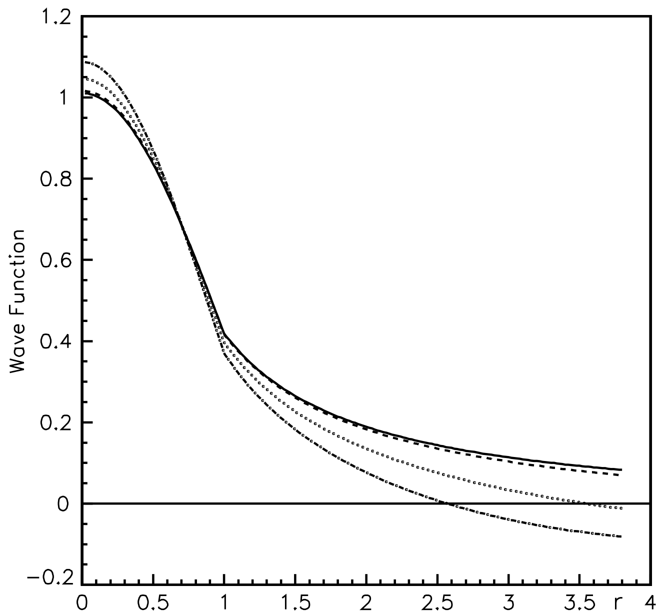

Fig. 1. Bound state wave function (solid line) and modified scattering state wave functions evaluated at different values of $k = 0.1, 0.4$ and 0.6 represented as dashed line, dotted line and dash-dotted line, respectively. $\lambda = 0.102$. The deviation at $r = 0$ is positive and increases with k.

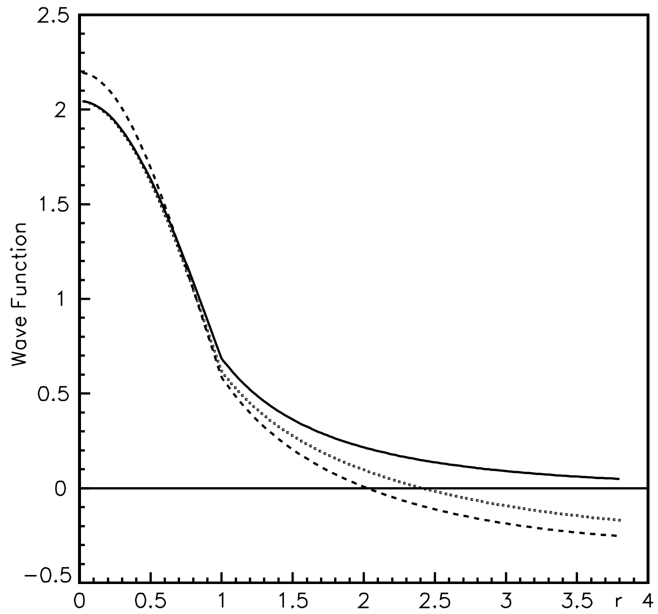

Fig. 2. Bound state wave function (solid line), modified scattering wave function (dashed line) evaluated at $k = 0.4$ and the corrected modified scattering wave function (dotted line). The deviation at the origin is positive. The corrected modified scattering function agrees very well with the bound state function. $\lambda = 0.463$ and $U_0 = 2.8$.

The exact solutions obtained allow us to study deviations from the extrapolation theorem analytically as one moves away from the bound state pole at $k = i\lambda$. This may be achieved by defining the ratio function

$$R(k,r) = -\sqrt{2\lambda(\lambda^2+k^2)}\,\frac{v(k,r)}{u_{\rm n}(r)}. \tag{5.4}$$

Since both the local and non-local parts of the potential have a finite range a, the ratio function defined above may be expanded as a power series in $(\lambda^2 + k^2)$ *viz.*,

$$R(k,r) = 1 + \sum_{p=1}^{\infty} R_p(r)(\lambda^2+k^2)^p. \tag{5.5}$$

In our case it may be shown that $R_1(r)$ takes the form

$$\begin{aligned} R_1(r) = {} & \frac{1}{\sin(Ka)} \\ & \times \left[\frac{\lambda a z^2 + [z^2 + \lambda a(\lambda a + A)(1-A)]\sin^2(Ka)}{(1+\lambda a)[z^2+\lambda a(\lambda a + A)]}\right]^{1/2} \\ & \times \left[\frac{a^2}{8z^2}\frac{[z^2+\lambda^2 a^2 - A][(1+\lambda a)(\lambda a + A) - z^2]}{(1+\lambda a)[z^2+\lambda a(\lambda a+A)]}\right. \\ & \left. + \frac{1}{z(1-A)K^2} - \frac{r^2}{6(1-A)} + O(r^4)\right], \end{aligned} \tag{5.6}$$

where

$$z = Ka(1-A). \tag{5.7}$$

6 Numerical calculations

In what follows we shall test the theorem numerically. For a non-local part $\rho = 1 - A = 0.5$, and a shallow local well of depth $U_0 = 2.2$, width $a = 1$, only one bound state with $\lambda = 0.102$ may be sustained. Figure 1 shows the bound state wave function, represented as a solid line in all the following figures, and the corresponding scattering wave function modified according to

$$v(k,r) \approx -\frac{1}{\sqrt{2\lambda(\lambda^2+k^2)}}u_{\rm n}(r), \tag{6.1}$$

which is evaluated at three different values $k = 0.1$ (dashed line), $k = 0.4$ (dotted line) and $k = 0.6$ (dash-dotted line). Obviously, the agreement is best for small values of k. At $r = 0$ the deviation is positive and increases with k. All the curves cross at the same value of $r \approx 0.7$. The cross-over point may easily be determined from (5.6) as it occurs when the first-order correction term vanishes. This crossing phenomena was also seen in the local case [4]. Further, the discontinuity in the derivatives at the boundary is clear.

When the local potential is reduced to -2.8, a single $1s$ bound state with $\lambda = 0.463$ is obtained. In fig. 2 the dashed line represents the scattering state wave function evaluated at $k = 0.4$. At $r = 0$, the modified scattering wave function lies above the bound state one. The dotted line represents the corrected wave function calculated to first order in $(\lambda^2 + k^2)$ using (5.6). For small r, the agreement is very good indicating that higher-order corrections are very small. To investigate the behavior away from the

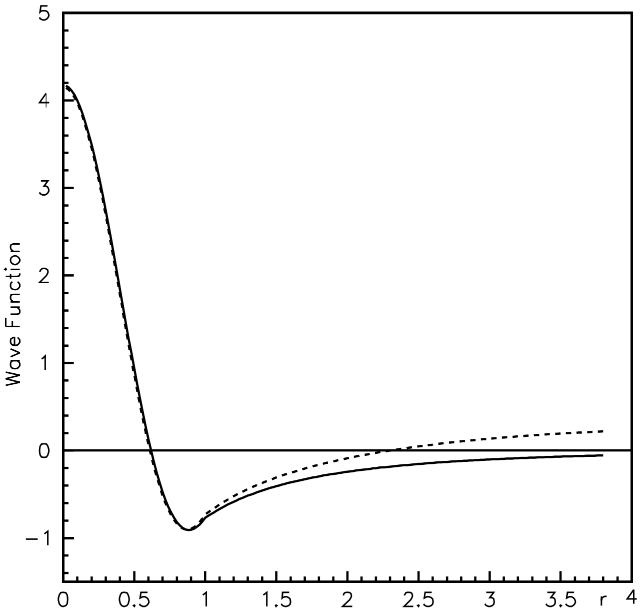

Fig. 3. Bound state wave function (solid line) and modified scattering state wave function (dashed line) evaluated at $k = 0.4$. A very good agreement is obtained corresponding to $\lambda = 0.463$ and $U_0 = 13.093$.

bound state pole, the scattering wave function is evaluated at different values of $k = 0.6$ and 0.8. The results showed that at the origin the deviation is positive and steadily increases with k as before. Further, all curves cross at the same point.

When the local potential is reduced even further $U_0 = 13.093$, a 2s bound state with the same value of $\lambda = 0.463$ is obtained. The results are shown in fig. 3 where the scattering wave function (dashed line) is evaluated at $k = 0.4$. A good agreement with the theory is obtained at short distances and the deviation is negative at the origin.

7 Discussion and conclusions

In the non-local case we have shown analytically that the relative normalisation of s-wave bound and corresponding scattering state wave functions is independent of the details of the potential at short distances, provided the energy is weak and the potential has a finite range. This has been accomplished in two ways. The first dealt with the non-local term directly, while the other transformed the potential into an equivalent local but energy-dependent one.

The theorem is tested analytically as well as numerically by solving the Schrödinger equation when the non-local potential took the form of a square well. An analytical expression for the deviation from the theory was also derived. The numerical resolution of the Schrödinger equation in the square well case shows that the theorem is valid at short distances and works best close to the bound state poles. The cross-over point, where all the curves intersect, was also seen in the local case [4], and can be easily predicted by setting the first-order correction in (5.6) to zero. A proof of the analyticity of the Jost solutions in the s-wave non-local case is presented.

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Appendix A. Analyticity of the Jost solutions in the non-local case

In this section we shall prove the analyticity of the Jost solutions so that the scattering function $v(k, r)$ in (2.2) may be analytically continued from the real axis into the complex k-plane to the positions of bound states.

As explained in sect. 4 using the transformation

$$v(k, r) = \frac{\chi(k, r)}{1 - \rho(r)}\ , \qquad \text{(A.1)}$$

eq. (2.2) may be transformed into

$$\chi''(k, r) + \left[k^2 - U_{\mathrm{e}}(k, r)\right] \chi = 0, \qquad \text{(A.2)}$$

where $U_{\mathrm{e}}(k, r)$ is an equivalent, effective, energy-dependent potential given by

$$U_{\mathrm{e}}(k, r) = -\frac{1}{(1 - \rho)} \left[\frac{\rho''}{2} - \frac{\rho'^{\,2}}{4(1 - \rho)} + \frac{\rho'}{r} - U(r) + \rho k^2\right], \qquad \text{(A.3)}$$

with an energy-dependent term $\rho k^2/(1 - \rho)$. We shall assume that $U(r)$ and $\rho(r)$ satisfy the conditions discussed in sects. 2.1 and 2.2. Hence

$$\chi(k, 0) = 0, \qquad \lim_{r\to\infty} \chi(k, r) = \frac{1}{k} \sin(kr + \delta_0(k)). \quad \text{(A.4)}$$

We shall introduce $\phi(k, r)$ as another regular solution of (A.2) satisfying the following boundary conditions:

$$\phi(k, 0) = 0, \qquad \phi'(k, 0) = 1 \qquad \text{(A.5)}$$

In what follows we follow closely the method presented by [10] for the local s-wave case and make amendments for the fact that the potential is energy dependent.

For k real, eq. (A.2) is real and the boundary conditions (A.5) are also real. Hence $\phi(k, r)$ is real and depends on k^2, consequently it is an even function of k. For complex values of k, the expression in the square brackets in (A.2) has no singularities at finite values of k, hence it is an entire function of k. According to the theorem of Poincaré [11], $\phi(k, r)$, which is a solution of (A.2) satisfying boundary conditions that do not depend on k, is an analytic function of k in the open complex k-plane. As $r \to \infty$ both $\rho(r)$ and $U(r)$ vanish and the function e^{-ikr} is a solution of (A.2). This function is identical to the asymptotic form of the Jost solution defined (asymptotically) as

$$\lim_{r\to\infty} f(k,r) = e^{-ikr}. \quad (A.6)$$

This defines $f(k,r)$ in the lower half of the complex k-plane. Introducing $f(k,r)$ as a solution for (A.2) we may write

$$f'' + k^2 f = U_{\rm e}(k,r) f. \quad (A.7)$$

With the boundary condition (A.6) the above differential equation may be transformed into a Volterra integral equation *viz.*,

$$g(k,r) = 1 + \int_r^\infty {\rm d}r'\ G_k(r'-r)\, U_{\rm e}(k,r')\, g(k,r'), \quad (A.8)$$

where

$$g(k,r) = e^{ikr} f(k,r), \quad (A.9)$$

and

$$G_k(x) \equiv \int_0^x {\rm d}y\ e^{-2iky}. \quad (A.10)$$

The solution of eq. (A.8) may be written as

$$g(k,r) = \sum_{n=0}^\infty g_n(k,r), \quad (A.11)$$

where $g_0 = 1$ and

$$g_n = \int_r^\infty {\rm d}r'\ G_k(r'-r)\, U_{\rm e}(k,r')\, g_{n-1}, \quad (A.12)$$

For Im$(k) < 0$, we shall show that the series (A.11) converges. In this case

$$|G_k(r'-r)| \le r' - r \le r'. \quad (A.13)$$

Therefore, the following conditions apply:

$$|g_1| \le p(r), \qquad p(r) \equiv \int_r^\infty {\rm d}r'\ r'\ |U_{\rm e}(k,r')|, \quad (A.14)$$

$$|g_2| \le \int_r^\infty {\rm d}r'\ r'\ |U_{\rm e}(k,r')| p(r') = \int_0^{p(r)} {\rm d}p\ p = \frac{p(r)^2}{2!}, \quad (A.15)$$

$$|g_3| \le \int_r^\infty {\rm d}r'\ r'\ |U_{\rm e}(k,r')| \frac{p(r)^2}{2!} = \frac{p(r)^3}{3!}, \dots . \quad (A.16)$$

Hence

$$|g_n| \le \frac{p(r)^n}{n!} \le \frac{p(0)^n}{n!}, \quad (A.17)$$

where

$$p(0) = \int_0^\infty {\rm d}r'\ r'\ |U_{\rm e}(k,r')|. \quad (A.18)$$

The series (A.11) converges uniformly provided $p(0) < \infty$. This is satisfied provided

$$\int_0^\infty {\rm d}r\, r\ |U(r)| < \infty, \qquad \int_0^\infty {\rm d}r\ \ r|\rho''(r)| < \infty. \quad (A.19)$$

To prove the analyticity of $g(k,r)$ with respect to k, we must show that the sequence of derivatives with respect to k also converges uniformly. Differentiating (A.11) and (A.12) with respect to k results in

$$|\dot g_n| = \int_r^\infty {\rm d}r'\ \left[\dot G\, U_{\rm e} + G\, \dot U_{\rm e}\right] |g_{n-1}| + \int_r^\infty {\rm d}r'\ G\ U_{\rm e} |\dot g_{n-1}|. \quad (A.20)$$

Noting that

$$|G_k(r-r')| < |\dot G_k(r-r')| \le (r'-r)^2 \le r'^2, \quad (A.21)$$

where the dot means differentiation with respect to k, one obtains

$$|\dot g_n| < \int_r^\infty {\rm d}r'\ r'^2\ \left[|U_{\rm e}(k,r')| + |\dot U_{\rm e}(k,r')|\right] |g_{n-1}| + \int_r^\infty {\rm d}r'\ r'^2\ |U_{\rm e}(k,r')|\ |\dot g_{n-1}|. \quad (A.22)$$

Hence

$$|\dot g_1| \le q(r),$$
$$q(r) \equiv \int_r^\infty {\rm d}r'\ r'^2\ \left[|U_{\rm e}(k,r')| + |\dot U_{\rm e}(k,r')|\right], \quad (A.23)$$

$$|\dot g_2| < \int_r^\infty {\rm d}r'\ r'^2\ \left[|U_{\rm e}(k,r')| + |\dot U_{\rm e}(k,r')|\right] |g_1| + \int_r^\infty {\rm d}r'\ r'^2\ |U_{\rm e}(k,r')|\ |\dot g_1|. \quad (A.24)$$

Using

$$|g_1| < |\dot g_1| \le q(r), \quad (A.25)$$

we may write

$$|\dot g_2| < 2 \int_r^\infty {\rm d}q\ \ q(r) = 2\, \frac{q(r)^2}{2!}. \quad (A.26)$$

Similarly,

$$|\dot g_3| < \int_r^\infty {\rm d}r'\ r'^2\ \left[|U_{\rm e}(k,r')| + |\dot U_{\rm e}(k,r')|\right] |g_2| + \int_r^\infty {\rm d}r'\ r'^2\ |U_{\rm e}(k,r')|\ |\dot g_2|. \quad (A.27)$$

Since

$$|g_2| < |\dot g_2| < 2\, \frac{q(r)^2}{2!}, \quad (A.28)$$

we have

$$|\dot{g}_3| < 4\,\frac{q(r)^3}{3!}. \tag{A.29}$$

Obviously, we may write the general term of the series of derivatives as

$$|\dot{g}_n| < 2^{(n-1)}\,\frac{q(r)^n}{n!}. \tag{A.30}$$

Defining

$$q(0) = \int_0^\infty \mathrm{d}r'\ r'^2\ \left[|U_\mathrm{e}(k,r')| + |\dot{U}_\mathrm{e}(k,r')|\right], \tag{A.31}$$

we have

$$|\dot{g}_n| < 2^{(n-1)}\frac{q(r)^n}{n!} < 2^{(n-1)}\frac{q(0)^n}{n!}. \tag{A.32}$$

Hence, the series

$$\dot{g}(k,r) = \sum_{n=0}^{\infty} \dot{g}_n(k,r) \tag{A.33}$$

converges uniformly, provided

$$\int_0^\infty \mathrm{d}r'\ r'^2\ \left[|U_\mathrm{e}(k,r')| + |\dot{U}_\mathrm{e}(k,r')|\right] < \infty. \tag{A.34}$$

Given that

$$\dot{U}_\mathrm{e}(k,r) = 2k\frac{\rho(r)}{1-\rho(r)}, \tag{A.35}$$

inequality (A.34) is satisfied if

$$\int_0^\infty \mathrm{d}r\, r^2\ |U(r)| < \infty, \qquad \int_0^\infty \mathrm{d}r\, r^2\ |\rho(r)| < \infty. \tag{A.36}$$

Hence we have shown that the Jost solution $f(k,r)$, which is related to $g(k,r)$ by (A.9), defined in the lower-half complex k-plane exists and is analytic for complex k when $\mathrm{Im}(k) < 0$ and is continuous along the real k-axis.

The region of analyticity of $f(k,r)$ may be extended into the upper half of the complex k-plane if we set the condition

$$\int_0^\infty \mathrm{d}r\, e^{mr}\ |U(r)| < \infty, \quad \int_0^\infty \mathrm{d}r\, e^{mr}\ |\rho(r)| < \infty, \tag{A.37}$$

where m is real and positive. In this case the function $f(k,r)$ is analytic for $\mathrm{Im}(k) < m/2$. It can also be shown that $f(k,r)$ has simple zeros that lie on the imaginary axis in the lower half of the complex k-plane, by following the method described in [10].

In the same way, it can be shown that $f(-k,r)$, which is defined asymptotically by

$$\lim_{r\to\infty} f(-k,r) = e^{ikr}, \tag{A.38}$$

is analytic in the upper half of the complex k-plane. If the conditions in (A.37) are satisfied, then $f(-k,r)$ is analytic for $\mathrm{Im}(k) > -m/2$, and has simple zeros that lie on the imaginary axis in the upper half of the complex k-plane.

The function $\phi(k,r)$ may be written as

$$\phi(k,r) = af(k,r) + bf(-k,r), \tag{A.39}$$

and the constants a and b may be determined using the boundary conditions (A.5), resulting in

$$\phi(k,r) = -\frac{1}{2ik}\left[f(-k)f(k,r) - f(k)f(-k,r)\right], \tag{A.40}$$

where

$$f(\pm k) = f(\pm k, 0). \tag{A.41}$$

It is not difficult to show that [2]

$$\chi(k,r) = \frac{1}{|f(k)|}\phi(k,r), \tag{A.42}$$

and by using (A.1), we have

$$v(k,r) = -\frac{1}{2ik}\frac{1}{\sqrt{1-\rho(r)}}\frac{1}{|f(k)|} \times \left[f(-k)f(k,r) - f(k)f(-k,r)\right]. \tag{A.43}$$

That is $v(k,r)$ is expressed in terms of a term depending on the non-local term $\rho(r)$ and a linear combination of Jost solutions.

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